MATH 543

# METHODS OF APPLIED MATHEMATICS I
## Final Exam

**January 9, 2015**
Friday 15.40-17.30

SOLUTIONS

**PROBLEMS**: Choose any three of the following four questions:

**1**[35]. Use the Poincare'-Lindstedt method to obtain a two term (first order perturbation) perturbation approximation to the following problem:

$$y'' + 9y = 3\varepsilon y^3, \ y(0) = 0, \ y'(0) = 1.$$

where $0 < \varepsilon << 1$.

**2**[35]. Use singular perturbation method to obtain a uniformly valid approximate solution. of the following the boundary value problem

$$\begin{aligned} \varepsilon y'' + y' &= 2t, \ 0 < t < 1, \ 0 < \varepsilon << 1, \\ y(0) &= 1, \ y(1) = 1, \end{aligned}$$

**3**[35]. Solve the following boundary value problem up to the first order in $\varepsilon$

$$\begin{aligned} y'' + (y')^2 + \varepsilon y &= 0, \ t > 0, \ 0 < \varepsilon << 1, \\ y(0) &= 0, \ y'(0) = 1, \end{aligned}$$

**4**[35]. Verify the following approximations for large $\lambda$.

(i) $\int_0^\infty e^{-\lambda t} \ln(1+t^2) dt \sim \frac{2!}{\lambda^3} - \frac{1}{2}\frac{4!}{\lambda^5} + \cdots, \quad \lambda >> 1,$
(ii) $\int_0^1 \sqrt{1+t}\, e^{\lambda(2t-t^2)}\, dt \sim \sqrt{\frac{\pi}{2\lambda}}\, e^{\lambda}. \quad \lambda >> 1,$
(iii) $\int_1^2 \sqrt{3+t}\, e^{\frac{\lambda}{t+1}}\, dt \sim \frac{8}{\lambda} e^{\lambda/2}, \quad \lambda >> 1$

**Appendix**: $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{a}$ where $a > 0$.

1. $y'' + 9y = 3\varepsilon y^3$, $t > 0$, $0 < \varepsilon << 1$.

$y(0) = 0$, $y'(0) = 1$

<u>SOLUTION</u>: Regular perturbation method produces secular terms. Hence we will apply the Poincaré-Lindstead method for this initial value problem. Let $\tau = \omega(\varepsilon) t$, then we get.

$$\omega^2 y_{\tau\tau} + 9y = 3\varepsilon y^3.$$

$$y(0) = 0, \quad \omega(\varepsilon) y_\tau(0) = 1.$$

$\varepsilon^0$: $\omega_0^2 y_{0\tau\tau} + 9y_0 = 0$, $\quad y_0(0) = 0$, $\omega_0 y_{0\tau}(0) = 1$.

$\varepsilon$ : $\omega_0^2 y_{1\tau\tau} + 9y_1 = 3y_0^2 - 2\omega_0\omega_1 y_{0\tau\tau}$, $\quad y_1(0) = 0$, $\omega_0 y_{1\tau}(0) + \omega_1 y_{0\tau}(0) = 0$

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Take $\omega_0 = 3$. Then we easily find $y_0(\tau) = \frac{1}{3}\sin\tau$ and $y_1$ satisfies

$$y_{1\tau\tau} + y_1 = \frac{1}{81}\sin^3\tau + \frac{2}{9}\omega_1 \sin\tau$$

$$= \frac{1}{4\cdot 81}(3\sin\tau - \sin 3\tau) + \frac{2}{9}\omega_1 \sin\tau$$

$$= \left(\frac{1}{4\cdot 27} + \frac{2}{9}\omega_1\right)\sin\tau - \frac{1}{4\cdot 81}\sin 3\tau$$

$$\omega_1 = -\frac{1}{8\cdot 3} = -\frac{1}{24} \Rightarrow \omega_1 = -\frac{1}{24}$$

Then we have

$$y_{1\tau\tau} + y_1 = -\frac{1}{4 \cdot 81} \sin 3\tau.$$

$$\omega_0 y_{1\tau}(0) + \omega_1 y_{0\tau}(0) = 0$$

$$y_1(0) = 0$$

General solution

$$y_1(\tau) = a\cos\tau + b\sin\tau + \frac{1}{32 \cdot 81}\sin 3\tau.$$

$y_1(0) = 0 \Rightarrow a = 0.$

$3y_{1\tau}(0) - \frac{1}{24} y_{0\tau}(0) = 0 \Rightarrow b = -\frac{1}{9 \cdot 32}.$

$$\Rightarrow y_{ap}(\tau) = \frac{1}{3}\sin\tau + \varepsilon\left[-\frac{1}{9 \cdot 32}\sin\tau + \frac{1}{32 \cdot 81}\sin 3\tau\right].$$

$$\omega(\varepsilon) = 3 - \frac{1}{24}\varepsilon \quad , \quad \tau = \left(3 - \frac{1}{24}\varepsilon\right)t$$

Residue function : $r(t,\varepsilon) = \omega^2 (y_{ap})_{\tau\tau} + 9y_{ap} - 3\varepsilon y_{ap}^3$

since $y_{ap}(\tau) = y_0 + \varepsilon y_1 + O(\varepsilon^2)$

$\Rightarrow |r(t,\varepsilon)| \leq \varepsilon^2 M_0(\varepsilon) \quad \forall t > 0$

There are no secular terms and $|\sin\tau| \leq 1$, $|\cos\tau| \leq 1$.

Then $\lim_{\varepsilon \to 0} r(t,\varepsilon) = 0$ when $M_0(\varepsilon)$ is positive

and bounded $\| M_0(\varepsilon) \| < M_1$ $(M_1 > 0)$.

(2) $\varepsilon y'' + y' = 2t, \quad y(0) = 1, \quad y(1) = 1.$

SOLUTION: a) outer solution: zeroth order perturbation ($t \sim 1$)

$y_0' = 2t \Rightarrow y_0(t) = t^2 + a$.

since $y(1) = 1. \Rightarrow a = 0$

$y_{out}(t) = t^2.$

b) Inner solution: ~~$t$~~ ($t \sim 1$).

$t = \delta \tau \Rightarrow \frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{1}{\delta} y_\tau = \delta \tau.$

$\delta = \varepsilon$ is the correct choice. $\Rightarrow$.

$y_{\tau\tau} + y_\tau = 2\varepsilon^2 \tau.$

Zeroth order perturbation

$y_0(\tau) = A + Be^{-\tau}$

since $y_0(0) = 1 \Rightarrow A + B = 1.$

$y_{in}(\tau) = A + (1 - A) e^{-t/\varepsilon}.$

3) Matching: $t = \sqrt{\varepsilon}\, \eta.$

$\lim_{\varepsilon \to 0} y_{out}(\sqrt{\varepsilon}\, \eta) = 0$

$\lim_{\varepsilon \to 0} y_{in}(\sqrt{\varepsilon}\, \eta) = A \quad \Rightarrow A = 0$

$y_{in} = e^{-t/\varepsilon}$

$\Rightarrow y_{ap}(t) = t^2 + e^{-t/\varepsilon}$

4) residue function $r(t,\varepsilon) = \varepsilon y_{ap}'' + y_{ap}' - 2t$

$$= 2\varepsilon.$$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} r(t,\varepsilon) = 0 \qquad \forall \; 0 < t < 1.$$

a uniformly valid approximation.

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3) $y'' + y'^2 + \varepsilon y = 0$ , $y(0) = 0$, $y'(0) = 1$

$t > 0$

SOLUTION: RPM.

$$y(t) = y_0(t) + \varepsilon y_1(t) + \dots$$

$\varepsilon^0$ : $y_0'' + y_0'^2 = 0$, $y_0(0) = 0$, $y_0'(0) = 1$

$\varepsilon$ : $y_1'' + 2y_0' y_1' + y_0 = 0$ , $y_1(0) = 0$, $y_1'(0) = 0$

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$$y_0'(t) = \frac{1}{t+1} \Rightarrow y_0(t) = \ln(1+t)$$

$$y_1'' + \frac{2}{t+1} y_1' = -\ln(1+t) \quad , y_1(0) = 0, \ y_1'(0) = 0$$

$$\Rightarrow y_1(t) = -\frac{1}{6}(1+t)^2 \ln(1+t) + \frac{5}{36}(1+t)^2 + \frac{1}{9(1+t)} - \frac{1}{4}$$

$$\Rightarrow \quad y_{ap} = \ln(1+t) + \varepsilon \left[ -\frac{1}{6}(1+t)^2 \ln(1+t) + \frac{5}{36}(1+t)^2 + \frac{1}{9(1+t)} - \frac{1}{4} \right]$$

residue function

$$r(t,\varepsilon) = y_{ap}'' + (y_{ap}')^2 + \varepsilon y_{ap}$$

$$\Rightarrow \ |r(t,\varepsilon)| \le \varepsilon^2 M(t,\varepsilon).$$

$M(t,\varepsilon)$ is bounded for $0 < t \le T < \infty$

for that interval $\lim_{\varepsilon \to 0} r(t,\varepsilon) = 0$ $\quad 0 < t \le T$.

otherwise not uniform.

4 a) $I = \int_0^\infty e^{-\lambda t} \ln(1+t^2)\, dt$

$$\int_0^{x} \frac{d\cancel{x}}{1+\cancel{x}} = \ln(1+x)$$

$\lambda t = u$

$$\frac{1}{\lambda}\int_0^\infty e^{-u} \ln\left(1+\frac{u^2}{\lambda^2}\right) du$$

$$\boxed{\int_0^x (1 - x + x^2 + = \ln(1+x) \quad = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \;, \; |x|<}$$

$$= \frac{1}{\lambda}\int_0^\infty e^{-u}\left(\frac{u^2}{\lambda^2} - \frac{1}{2}\frac{u^4}{\lambda^4} + \dots\right) du$$

$$= \frac{1}{\lambda}\int_0^\infty e^{-u}\left(\frac{u^2}{\lambda^2}\right) du - \frac{1}{2\lambda^5}\int e^{-u} u^4\, du$$

$$= \frac{1}{\lambda^3}\left(\int_0^\infty e^{-u} u^2\, du\right) - \frac{1}{2\lambda^5}\left(\int_0^\infty e^{-u} u^4\, du\right)$$

$$\int e^{-u} u^2\, du = -u^2 e^{-u}\Big|_0^\infty + \int e^{-u} 2u\, du$$

$$= 2\left(-e^{-u} u\Big|_0^\infty + \cancel{0}(-e^{-u})\Big|_0^\infty\right) = 2$$

$$\int e^{-u} u^4\, du = 4 \cdot 3 \cdot 2 \quad \dots$$

$$I = \frac{2}{\lambda^3} - \frac{4!}{2}\frac{1}{\lambda^5}$$

4b) $\int_0^1 \sqrt{1+t}\, e^{\lambda(2t-t^2)}\, dt$ Lecture Notes

4c) Lecture Note